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Linear Programming, the Global Approach

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INTRODUCTION

Many optimization problems require the maximizing or minimizing of a linear function subject to linear constraints. Such problems, called linear programs (l.p.'s), were intensively investigated in the years after WWII and a beautiful theory developed. The basic results of this theory, such as l.p. duality and the simplex algorithm, have become standard fare for combinatorial mathematicians. (Cf., Hall's book [4], or the more recent one by Lawler [7].)

In this paper we shall investigate l.p.'s from the point of view of category theory. We first ask whether l.p.'s and related problems have morphisms, i.e., transformations which preserve the defining characteristics of the problem. If the answer is yes, we then ask what structure the resulting category has and what that implies about the problem. For the definitions and results of category theory which we use here, the reader is referred to MacLane's book [8].

1. STANDARD DEFINITIONS

A standard version of the l.p. problem is (in matrix notation):

$$\begin{aligned} \text{maximize} \quad & cx \\ \text{subject to} \quad & Ax \leqslant b, \\ & x \geqslant 0. \end{aligned}$$

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A denotes an $m \times n$ matrix of real numbers, b a column-vector and c a row-vector. The inequalities are defined coordinatewise.

EXAMPLE 1. Maximize $x_1 + x_2 + x_3 + x_4$ subject to

$$\begin{aligned} x_1 \qquad\quad + 2x_3 + \ \ x_4 &\leqslant 3, \\ x_2 + \ \ x_3 + 2x_4 &\leqslant 3, \\ 2x_1 - \ \ x_2 + 5x_3 - 3x_4 &\leqslant 1, \\ -x_1 + 2x_2 - 3x_3 + 5x_4 &\leqslant 1, \end{aligned}$$

x_1, x_2, x_3 and $x_4 \geqslant 0$. In this example

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & 5 & -3 \\ -1 & 2 & -3 & 5 \end{pmatrix},$$

$$b = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

and

$$c = (1 \quad 1 \quad 1 \quad 1).$$

The instance of the l.p. problem which A, b and c characterize is denoted by $P = (A, b, c)$. If x satisfies the constraints $Ax \leqslant b$ and $x \geqslant 0$, it is called *P-feasible*, and if it maximizes cx over all P-feasible vectors, it is called *P-optimal.* If x is P-optimal, then $cx = v(P)$ is called *the value of P*. If P has no feasible solutions, then $v(P) = -\infty$.

2. MORPHISMS FOR LINEAR PROGRAMS

Now we address the main question of this paper: What transformations on l.p.'s preserve their solutions and values? In particular, when can a l.p. be reduced to a simpler one, P'? This can be done, for instance, if one of the inequalities of P is implied by the others; eliminating it would not alter the set of feasible solutions. We wish though to make a general definition of such reductions and study them systematically.

DEFINITION. A *demireduction* $\varphi: P \to P'$ consists of a pair of matrices $\varphi = (\varphi_l, \varphi_r)$, $\varphi_l, \varphi_r \geqslant 0$, such that

(i) $\varphi_l A \geqslant A' \varphi_r$,
(ii) $\varphi_l b \leqslant b'$ and
(iii) $c \leqslant c' \varphi_r$.

Note that φ_l is $m' \times m$ and φ_r is $n' \times n$.

EXAMPLE 2a. For any l.p. P, the identity $\iota_p = (I_m, I_n)$, where I_m is the $m \times m$ unit matrix, is a demireduction $\iota_p: P \to P$.

EXAMPLE 2b. Observe that the l.p. of Example 1 has a symmetry which interchanges x_1 with x_2, and x_3 with x_4. Also it interchanges the first pair of inequalities as well as the second pair. This symmetry is a demireduction $\varphi: P \to P$ with

$$\varphi_l = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \varphi_r.$$

EXAMPLE 2c. Removing an inequality gives a demireduction: Suppose the first inequality is to be removed from $P = (A, b, c)$ to give $P' = (A', b', c')$, i.e.,

$$A = \begin{pmatrix} A_1 \\ A' \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b' \end{pmatrix}.$$

This determines $\varphi: P \to P'$ where

$$\varphi_l = (0 \quad I_{m-1})$$

and

$$\varphi_r = I_n.$$

THEOREM 1. *If $\varphi: P \to P'$ is a demireduction, then for all P-feasible x, $x' = \varphi_r x$ is P'-feasible and $cx \leqslant c'x'$. Therefore $v(P') \geqslant v(P)$.*

Proof. x being P-feasible means

$$Ax \leqslant b,$$
$$x \geqslant 0.$$

Since $x \geqslant 0$ and $\varphi_r \geqslant 0$, $x' = \varphi_r x \geqslant 0$. Also $\varphi_l \geqslant 0$ and $Ax \leqslant b$ implies

$$\varphi_l(Ax) \leqslant \varphi_l b \leqslant b'.$$

But $\varphi_l(Ax) = (\varphi_l A)x \geqslant (A'\varphi_r)x = A'(\varphi_r x) = A'x'$. Thus $A'x' \leqslant b'$ and x' is P'-feasible. Furthermore $c'x' = c'(\varphi_r x) = (c'\varphi_r)x \geqslant cx$. End of proof.

Note. The definition of demireduction was made as general as possible, but so that the proof of Theorem 1 would still hold.

3. The Categories of Linear Programming

We form the category *LPD* whose objects are l.p.'s and whose morphisms are demireductions:

If $\varphi: P \to P'$ and $\tau: P' \to P''$, then $\tau \circ \varphi$: $P \to P''$ has components $(\tau \circ \varphi)_l = \tau_l \varphi_l$ and $(\tau \circ \varphi)_r = \tau_r \varphi_r$, i.e., composition defined by componentwise (matrix) multiplication. To verify that this composition is well defined, note that $(\tau \circ \varphi)_l \geqslant 0$, $(\tau \circ \varphi)_r \geqslant 0$ and $(\tau \circ \varphi)_l A = (\tau_l \varphi_l) A = \tau_l(\varphi_l A) \geqslant \tau_l(A'\varphi_r) = (\tau_l A')\varphi_r \geqslant (A''\tau_r)\varphi_r = A''(\tau_r \varphi_r) = A''(\tau \circ \varphi)_r$. Also, $(\tau \circ \varphi)_l b = (\tau_l \varphi_l) b = \tau_l(\varphi_l b) \leqslant \tau_l b' \leqslant b''$, and similarly $c''(\tau \circ \varphi)_r \geqslant c$. Associativity of this composition follows from that of matrix multiplication and similarly for the identities (Example 2a).

A fully satisfying notion of reduction for l.p.'s should give the essential equivalence of P and P'. Defining it to be a pair of demireductions $\varphi: P \to P'$ and $\rho: P' \to P$ would suffice, but an ultimately more useful definition is as follows: *a (full) reduction* (φ, ρ): $P \to P'$ consists of a demireduction $\varphi: P \to P'$ and a right inverse $\rho: P' \to P$; i.e., ρ is a demireduction and $\varphi \circ \rho = \iota_{P'}$.

Examples 3a and 3b. The demireductions of Examples 2a and 2b are self-inverse and therefore extend to (full) reductions.

Example 3c. In Example 2c, the demireduction (given by removing the first inequality $A_1 x \leqslant b$) has a right inverse if that first inequality is implied by the others: If there exists a vector $\alpha \geqslant 0$ such that $\alpha A' \geqslant A_1$ and $\alpha b' \leqslant b_1$ then $\rho: P' \to P$ is defined by

$$\rho_l = \begin{pmatrix} \alpha \\ I_{m-1} \end{pmatrix}$$

and

$$\rho_r = I_n$$

is a right inverse for the φ of Example 2c.

Note a. If (φ, ρ): $P \to P'$ is a reduction, then $m \geqslant m'$ and $n \geqslant n'$ since $\varphi_l \rho_l = I_{m'}$ and $\varphi_r \rho_r = I_{n'}$ imply that φ_l and φ_r are of rank at least m' and n', respectively.

Note b. If (φ, ρ): $P \to P'$ is a reduction, then $\rho_l b' \leqslant b$ so $\varphi_l(\rho_l b') \leqslant \varphi_l b \leqslant b'$. But $\varphi_l(\rho_l b') = (\varphi_l \rho_l) b' = I_{m'} b' = b'$. Therefore $b' = \varphi_l b$ and by duality (see the following section), $c' = c\rho_r$. From Example 3c, however, it is apparent that the inequalities $\varphi_l A \geqslant A' \varphi_r$ and $\rho_l b' \leqslant b$ and thus their duals, $\rho_l A' \geqslant A \rho_r$ and $c \leqslant c' \varphi_r$, need not be equalities. Actually, the relationship between these inequalities is not totally clear to the author at this time; it appears though that the inequalities $\varphi_l b \leqslant b'$ and $c \leqslant c' \varphi_r$ in the definition of a demireduction are not essential. Since this restriction would facilitate the development of the theory in several places, we shall make the restriction. That is, the definition of a *demireduction* $\varphi: P \to P'$ will be a pair of matrices $\varphi_l, \varphi_r \geqslant 0$ such that

(i) $\varphi_l A \geqslant A' \varphi_r$

(ii) $\varphi_l b = b'$ and

(iii) $c = c' \varphi_r$.

We may now define our second category LP whose objects are l.p.'s and whose morphisms are (full) reductions with composition defined componentwise. It is easily verified that this composition of reductions gives a reduction, that it is associative and we have already observed that identities are reductions (Examples 3a and 3b).

4. Duality

The *dual* of $P = (A, b, c)$ is $P^* = (-A^t, -c^t, -b^t)$. Note that $(P^*)^* = P$. P^* may be simplified somewhat as

$$\begin{array}{ll} \text{minimize} & ub \\ \text{subject to} & uA \geqslant c, \\ & u \geqslant 0. \end{array}$$

One of the pillars of the theory of l.p.'s is

THE DUALITY THEOREM. *If x is P-feasible and u^t is P^*-feasible, then $ub \geqslant cx$. If they are both optimal, then $ub = cx$, i.e., $v(P^*) = -v(P)$.*

THEOREM 2. *Duality of l.p.'s induces a contravariant functor $*: LPD \to LPD$ and a covariant functor $*: LP \to LP$.*

Proof. In both categories $*_{\text{ob}}(P) = P^*$. If $\varphi: P \to P'$ is a demireduction, then $*_{\text{Hom}}(\varphi) = \varphi^* = (\varphi_r^t, \varphi_l^t)$. $\varphi^*: P'^* \to P^*$ since $\varphi_r^t, \varphi_l^t \geqslant 0$ and

(i) $\varphi_l A \geqslant A' \varphi_r$ implies $\varphi_r^t(-A'^t) \geqslant (-A^t)\,\varphi_l^t$,

(ii) $c = c'\varphi_r$ implies $\varphi_r^t(-c'^t) = -c^t$,

(iii) $\varphi_l b = b'$ implies $(-b')^t = (-b^t)\,\varphi_l^t$.

Also

$$(*_{\text{Hom}}(\varphi \circ \tau))_l = ((\varphi \circ \tau)_r)^t = (\varphi_r \tau_r)^t = \tau_r^t \varphi_r^t = (*_{\text{Hom}}(\tau) \circ *_{\text{Hom}}(\varphi))_l$$

and similarly with the r-component of $\varphi \circ \tau$. Thus $*_{\text{Hom}}(\varphi \circ \tau) = *_{\text{Hom}}(\tau) \circ *_{\text{Hom}}(\varphi)$. Furthermore $*_{\text{Hom}}(\iota_p) = (I_n, I_m) = \iota_{p^*}$.

On the other hand, if $(\varphi, \rho): P \to P'$ is a (full) reduction, then $*_{\text{Hom}}(\varphi, \rho) = (\rho^*, \varphi^*)$. $(\rho^*, \varphi^*): P^* \to P'^*$ since $\rho^*: P^* \to P'^*$ and $\varphi^*: P'^* \to P^*$ are demireductions and $\rho^* \circ \varphi^* = (\varphi \circ \rho)^* = (\iota_{p'})^* = \iota_{p'^*}$. Also,

$$\begin{aligned} *_{\text{Hom}}((\tau, \delta) \circ (\varphi, \rho)) &= *_{\text{Hom}}(\tau \circ \varphi, \rho \circ \delta) \\ &= ((\rho \circ \delta)^*, (\tau \circ \varphi)^*) \\ &= (\delta^* \circ \rho^*, \varphi^* \circ \tau^*) \\ &= (\delta^*, \tau^*) \circ (\rho^*, \varphi^*) \\ &= *_{\text{Hom}}(\tau, \delta) \circ *_{\text{Hom}}(\varphi, \rho) \end{aligned}$$

and lastly,

$$\begin{aligned} *_{\text{Hom}}(\iota_p, \iota_p) &= (\iota_p^*, \iota_p^*) \\ &= (\iota_{p^*}, \iota_{p^*}). \end{aligned}$$

End of proof.

5. An "Equivalent" Form of the Linear Programming Problem

Let $\overline{LPD}$ be the category whose objects are l.p.'s in *equality form*, i.e., $\bar{P} = (\bar{A}, \bar{b}, \bar{c})$ represents the problem

$$\begin{aligned} \text{maximize} \quad & \bar{c}x \\ \text{subject to} \quad & \bar{A}x = \bar{b}, \\ & \bar{x} \geqslant 0. \end{aligned}$$

A morphism $\bar{\varphi}: \bar{P} \to \bar{P}'$ in $\overline{LPD}$ consists of a pair of matrices $\bar{\varphi} = (\bar{\varphi}_l, \bar{\varphi}_r)$, $\bar{\varphi}_l$ arbitrary and $\bar{\varphi}_r \geqslant 0$, such that

(i) $\bar{\varphi}_l \bar{A} = \bar{A}' \bar{\varphi}_r$,

(ii) $\bar{\varphi}_l \bar{b} = \bar{b}'$ and

(iii) $\bar{c} = \bar{c}' \bar{\varphi}_r$.

All standard definitions for $\overline{LPD}$ are essentially the same as for LPD and $\overline{LPD}$ is to $\overline{LPD}$ as LP is to LPD.

THEOREM 3. *If $\bar{\varphi}: \bar{P} \to \bar{P}'$ is an $\overline{LPD}$-morphism, then for all $\bar{P}$-feasible x, $x' = \bar{\varphi}_r x$ is $\bar{P}'$-feasible and $\bar{c}x \leqslant \bar{c}'x'$. Therefore $v(\bar{P}') \geqslant v(\bar{P})$.*

Proof. Essentially the proof is the same as that for Theorem 1.

EXAMPLE 4. Given a set of basic variables for $\bar{P}$, the transformation of $\bar{P}$ to canonical form (see [1, Sections 4-2 and 8-1]) is an $\overline{LP}$-morphism. More explicitly, if the basic variables correspond to the first m columns of $\bar{A}$, then we may write $\bar{A} = (BA_0)$, B invertible. Then

$$\begin{aligned} \bar{A}' &= B^{-1}\bar{A} = (I_m[B^{-1}A_0]), \\ b' &= B^{-1}\bar{b}, \\ \bar{c}' &= \bar{c} \end{aligned}$$

represents the canonical form. Therefore $\bar{\varphi}: \bar{P} \to \bar{P}'$ has $\varphi_l = B^{-1}$, $\varphi_r = I_n$ with $\bar{\rho}_l = B$, $\bar{\rho}_r = I_n$ giving the right (and left) inverse. In particular, the pivot steps of the simplex algorithm are $\overline{LP}$-isomorphisms. They are relatively easy to compute and each one decreases the value of $\bar{c}x$ for the corresponding basic solution (see Chaps. 5 and 8 of [1]).

The classical operation of adjoining slack variables is used to transform a l.p. in inequality form to an "equivalent" one in equality form. This operation may be extended to a functor $S: LPD \to \overline{LPD}$:

$$S_{\text{ob}}(P) = \bar{P} = (\bar{A}, \bar{b}, \bar{c}),$$

where

$$\begin{aligned} \bar{A} &= (I_m A), \\ \bar{b} &= b, \\ \bar{c} &= (Oc). \end{aligned}$$

If $\varphi: P \to P'$ in LPD, then $S_{\mathrm{Hom}}(\varphi) = \bar{\varphi}: \bar{P} \to \bar{P}'$, where $\bar{\varphi} = (\bar{\varphi}_l, \bar{\varphi}_r)$ is defined by $\bar{\varphi}_l = \varphi_l$ and

$$\bar{\varphi}_r = \begin{pmatrix} \varphi_l & [\varphi_l A - A'\varphi_r] \\ 0 & \varphi_r \end{pmatrix} \geqslant 0.$$

To see that in fact $\bar{\varphi}: \bar{P} \to \bar{P}'$ note that

$$\bar{\varphi}_l \bar{A} = \varphi_l (I_m A) = (\varphi_l [\varphi_l A])$$

and

$$\begin{aligned} \bar{A}'\bar{\varphi}_r &= (I_{m'} A') \begin{pmatrix} \varphi_l & [\varphi_l A - A'\varphi_r] \\ 0 & \varphi_r \end{pmatrix} \\ &= (\varphi_l [\varphi_l A - A'\varphi_r + A'\varphi_r]) \\ &= (\varphi_l [\varphi_l A]). \end{aligned}$$

Therefore $\bar{\varphi}_l \bar{A} = \bar{A}'\bar{\varphi}_r$. Also, $\bar{\varphi}_l \bar{b} = \varphi_l b = b' = \bar{b}'$ and

$$\begin{aligned} \bar{c}'\bar{\varphi}_r &= (Oc') \begin{pmatrix} \varphi_l & [A'\varphi_r - \varphi_l A] \\ 0 & \varphi_r \end{pmatrix} \\ &= (O[c'\varphi_r]) \\ &= (Oc) = \bar{c}. \end{aligned}$$

To verify that S is a functor, note that

$$\begin{aligned} S(\iota_p) &= \left(I_m, \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix} \right) = (I_m, I_{m+n}) \\ &= \iota_{S(P)}. \end{aligned}$$

Also if $\varphi: P \to P'$, $\tau: P' \to P''$ then

$$\begin{aligned} S(\tau \circ \varphi)_l &= (\tau \circ \varphi)_l = \tau_l \varphi_l = (S(\tau))_l \, (S(\varphi))_l \\ &= (S(\tau) \circ S(\varphi))_l \end{aligned}$$

and

$$\begin{aligned} S(\tau \circ \varphi)_r &= \begin{pmatrix} \tau_l \varphi_l & [\tau_l \varphi_l A - A''\tau_r \varphi_r] \\ 0 & \tau_r \varphi_r \end{pmatrix} \\ &= \begin{pmatrix} \tau_l & [\tau_l A' - A''\tau_r] \\ 0 & \tau_r \end{pmatrix} \begin{pmatrix} \varphi_l & [\varphi_l A - A'\varphi_r] \\ 0 & \varphi_r \end{pmatrix} \\ &= (S(\tau) \circ S(\varphi))_r. \end{aligned}$$

Therefore $S(\tau \circ \varphi) = S(\tau) \circ S(\varphi)$.

On the otherhand suppose there is a functor $T\colon \overline{LPD} \to LPD$ based on the standard trick of representing an equality by two inequalities:

$$T_{\text{ob}}(\bar{P}) = P = (A, b, c),$$

where

$$A = \begin{pmatrix} \bar{A} \\ -\bar{A} \end{pmatrix}$$

$$b = \begin{pmatrix} \bar{b} \\ -\bar{b} \end{pmatrix}$$

and

$$c = \bar{c}.$$

Then if $\bar{\varphi}\colon \bar{P} \to \bar{P}'$, $\bar{\varphi} = (\bar{\varphi}_l, \bar{\varphi}_r)$, it seems natural to define $T_{\text{Hom}}(\bar{\varphi}) = \varphi\colon P \to P'$ by

$$\varphi_l = \begin{pmatrix} \bar{\varphi}_l^+ & \bar{\varphi}_l^- \\ \bar{\varphi}_l^- & \bar{\varphi}_l^+ \end{pmatrix},$$

where

$$\begin{aligned} (\bar{\varphi}_l^+)_{ij} &= (\bar{\varphi}_l)_{ij} && \text{if} \quad (\bar{\varphi}_l)_{ij} \geqslant 0 \\ &= 0 && \text{otherwise} \\ (\bar{\varphi}_l^-)_{ij} &= -(\bar{\varphi}_l)_{ij} && \text{if} \quad (\bar{\varphi}_l)_{ij} \leqslant 0 \\ &= 0 && \text{otherwise,} \end{aligned}$$

and $\varphi_r = \bar{\varphi}_r$. By definition then $\varphi_l, \varphi_r \geqslant 0$ and

$$\begin{aligned} \varphi_l A &= \begin{pmatrix} \bar{\varphi}_l^+ & \bar{\varphi}_l^- \\ \bar{\varphi}_l^- & \bar{\varphi}_l^+ \end{pmatrix} \begin{pmatrix} \bar{A} \\ -\bar{A} \end{pmatrix} \\ &= \begin{pmatrix} [\bar{\varphi}_l \bar{A}] \\ [-\bar{\varphi}_l \bar{A}] \end{pmatrix} = \begin{pmatrix} [\bar{A}' \bar{\varphi}_r] \\ [-\bar{A}' \bar{\varphi}_r] \end{pmatrix} = A' \varphi_r. \end{aligned}$$

Also

$$\begin{aligned} \varphi_l b &= \begin{pmatrix} \bar{\varphi}_l^+ & \bar{\varphi}_l^- \\ \bar{\varphi}_l^- & \bar{\varphi}_l^+ \end{pmatrix} \begin{pmatrix} \bar{b} \\ -\bar{b} \end{pmatrix} = \begin{pmatrix} [\bar{\varphi}_l \bar{b}] \\ [-\bar{\varphi}_l \bar{b}] \end{pmatrix} = \begin{pmatrix} \bar{b}' \\ -\bar{b}' \end{pmatrix} \\ &= b' \end{aligned}$$

and

$$c'\varphi_r = \bar{c}'\bar{\varphi}_r = \bar{c} = c.$$

Therefore $\varphi: P \to P'$. In order to verify that T is a functor, we note that

$$T(\iota_{\bar{P}}) = \left(\begin{pmatrix} I_m & 0 \\ 0 & I_m \end{pmatrix}, I_n \right) = \iota_{T(\bar{P})}.$$

However, if $\bar{\varphi}: \bar{P} \to \bar{P}'$, $\bar{\tau}: \bar{P}' \to \bar{P}''$, then

$$\begin{aligned}(T(\bar{\tau} \circ \bar{\varphi}))_l &= \begin{pmatrix} [\bar{\tau}_l \bar{\varphi}_l]^+ & [\bar{\tau}_l \bar{\varphi}_l]^- \\ [\bar{\tau}_l \bar{\varphi}_l]^- & [\bar{\tau}_l \bar{\varphi}_l]^+ \end{pmatrix} \\ &\neq \begin{pmatrix} \bar{\tau}_l^+ & \bar{\tau}_l^- \\ \bar{\tau}_l^- & \bar{\tau}_l^+ \end{pmatrix} \begin{pmatrix} \bar{\varphi}_l^+ & \bar{\varphi}_l^- \\ \bar{\varphi}_l^- & \bar{\varphi}_l^+ \end{pmatrix} \\ &= (T(\bar{\tau}) \circ T(\bar{\varphi}))_l\end{aligned}$$

and so T is not a functor!

The conclusion to be drawn from this fiasco seems to be that $\overline{LP}$ is stronger (has more reductions) than LP. This follows from the fact that if $(\bar{\varphi}, \bar{\rho})$: $\bar{P} \to \bar{P}'$, then $T(\bar{\varphi} \circ \bar{\rho}) \neq T(\bar{\varphi}) \circ T(\bar{\rho})$, $T(\bar{\varphi} \circ \bar{\rho}) = T(\iota_{p'}) = \iota_{T(p')}$. So T does not even give a Hom-function on $\overline{LP}$ to LP. This "explains" the importance of slack variables in l.p. theory.

6. Matrix Games

A *matrix game* is determined by a matrix A (see [1, Chap. 13]). The *von Neumann minimax value* of the game is

$$\begin{aligned} m(A) &= \min_q \max_p (pAq) \\ &= \max_p \min_q (pAq), \end{aligned}$$

where p and q range over all strategy vectors for the row and column player, respectively. That is $p \geqslant 0$ and $p1_m = 1$, where 1_m is a column vector of dimension m whose every entry is one. Also $q \geqslant 0$, $1_n q = 1$, where 1_n is a row vector of dimension n whose every entry is one.

Note. A standard simplification of the above definition gives

$$\begin{aligned} m(A) &= \min\{k: \forall q, Aq \leqslant k1_m\} \\ &= \max\{k: \forall p, pA \geqslant k1_n\}. \end{aligned}$$

Raising the question of morphisms for this new problem, we obtain the following definition: A *demigamemorphism* $\varphi: A \to A'$ consists of a pair of matrices $\varphi = (\varphi_l, \varphi_r)$, $\varphi_l, \varphi_r \geqslant 0$, such that

(i) $\varphi_l A \geqslant A'\varphi_r$,

(ii) $\varphi_l 1_m = 1_{m'}$,

(iii) $1_n = 1_{n'}\varphi_r$.

THEOREM 4. *If $\varphi: A \to A'$ is a demigamemorphism, then for every column strategy, q, of A such that $Aq \leqslant k1_m$, $q' = \varphi_r q$ is a column strategy for A' such that $A'q' \leqslant k1_{m'}$. Therefore $m(A') \leqslant m(A)$. Furthermore for every row strategy p' of A' such that $p'A' \geqslant k1_{n'}$, $p = p'\varphi_l$ is a row strategy for A such that $pA \geqslant k1_n$.*

Proof. $Aq \leqslant k1_m$ implies $\varphi_l(Aq) \leqslant \varphi_l(k1_m) = k1_{m'}$, and $\varphi_l(Aq) = (\varphi_l A)q \geqslant (A'\varphi_r)q = A'(\varphi_r q) = A'q'$. Therefore $A'q' \leqslant k1_{m'}$. Also, $q' = \varphi_r q \geqslant 0$ and $1_{n'}q' = 1_{n'}(\varphi_r q) = (1_{n'}\varphi_r)q = 1_n q = 1$. The proof of the statement for row strategies is essentially the same.

As with l.p.'s, we define a (full) gamemorphism $(\varphi, \rho): A \to A'$ to be a pair of demigamemorphisms $\varphi: A \to A'$ and $\rho: A' \to A$ such that $\varphi \circ \rho = \iota_{A'}$. We again have a pair of categories; *MGD* having matrix games as objects and demigamemorphisms as morphisms and *MG* having the same objects but (full) gamemorphisms as morphisms.

Given a game with matrix A, associate with it a l.p. $P_{A,+,+} = (A, 1_m, 1_n)$.

Note. The function $F(A) = P_{A,+,+}$ is the object function of functors which imbed *MGD* in *LPD* and *MG* in *LP*. The Hom-functions that go with them are just the identities $F(\varphi) = \varphi$ and $F(\varphi, \rho) = (\varphi, \rho)$.

THEOREM 5.

$$m(A) = \frac{1}{v(P_{A,+,+})} \qquad \text{if} \quad 0 < m(A) < \infty$$

$$= \frac{1}{v(P_{A,-,-})} \qquad \text{if} \quad -\infty < m(A) < 0,$$

where $P_{A,-,-} = (A, -1_m, -1_n)$.

Proof. If $0 < m(A) < \infty$, then

$$m(A) = \min\{k > 0 \mid Aq \leqslant k1_m,\ q \geqslant 0,\ 1_n q = 1\}$$

$$= \min\left\{k > 0 \,\middle|\, A\left[\frac{q}{k}\right] \leqslant 1_m,\ \frac{q}{k} \geqslant 0,\ 1_n\left[\frac{q}{k}\right]\right) = \frac{1}{k}\right\}$$

$$= \min\left\{\frac{1}{1_n x} \quad Ax \leqslant 1_m,\ x \geqslant 0\right\}$$

$$= \frac{1}{\max\{1_n x \mid Ax \leqslant 1_m,\ x \geqslant 0\}} = \frac{1}{v(P_{A,+,+})}.$$

Similarly if $-\infty < m(A) < 0$, then

$$\begin{aligned} m(A) &= \min\{k < 0 \mid Aq \leqslant k 1_n,\ q \geqslant 0,\ 1_n q = 1\} \\ &= -\max\{k > 0 \mid Aq \leqslant -k 1_n,\ q \geqslant 0,\ 1_n q = 1\} \\ &= -\frac{1}{\min\{1_n x \mid Ax \leqslant -1_n,\ x \geqslant 0\}} \\ &= \frac{1}{\max\{(-1_n) x \mid Ax \leqslant -1_n,\ x \geqslant 0\}} \\ &= \frac{1}{v(P_{A,-,-})}. \end{aligned}$$

Game-programs, i.e., those l.p.'s of the form $P_{A,\pm,\pm}$, appear to be of quite special form: If $P = (A, b, c)$, where all entries of b and c are positive, then

$$(\varphi, \rho)\colon P \to P_{A',+,+}$$

by

$$\varphi_l = \begin{bmatrix} \frac{1}{b_1} & & & 0 \\ & \frac{1}{b_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{b_m} \end{bmatrix}, \qquad \varphi_r = \begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_n \end{pmatrix},$$

$$\rho_l = \begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_m \end{pmatrix}, \qquad \rho_r = \begin{bmatrix} \frac{1}{c_1} & & & 0 \\ & \frac{1}{c_2} & & \\ & & \ddots & \\ & & & \frac{1}{c_m} \end{bmatrix}$$

and $A' = \varphi_l A \rho_r$. The same idea can be used to reduce an arbitrary P to a P'

where b' and c' have entries only ± 1 and 0. Reduction of P to a game-program, however, appears doubtful.

The category of matrix games has a simple symmetry induced by adding a constant to each entry of very matrix: For fixed k, let $K_k(A) = A + (k)$ and $K(\varphi) = \varphi$. Then $K_k: MG \to MG$ is a functor and $K_k^{-1} = K_{-k}$. Also $m(K_k(A)) = m(A) + k$.

Note. Another special property of games, and their corresponding programs, is that solutions give morphisms: If p and q are optimal strategies for a game A, then

$$(\varphi, \rho)\colon A \to A' = (m(A))$$

is defined by

$$\varphi_l = p, \qquad \varphi_r = 1_n,$$
$$\rho_l = 1_m, \qquad \rho_r = q.$$

This observation leads us to a theorem of Gale *et al.* [3].

THEOREM 6. *If the matrix A of a game can be partitioned into blocks (i.e., with rows and columns partitioned)*

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n'} \\ A_{21} & A_{22} & \cdots & \\ \vdots & & & \\ A_{m'1} & & \cdots & A_{m'n'} \end{pmatrix}$$

and there exist sets of strategies $\{p_{i'}\}_{1 \leqslant i' \leqslant m'}$ and $\{q_{j'}\}_{1 \leqslant j' \leqslant n'}$ such that for all i', j', $p_{i'}$ and $q_{j'}$ are optimal for $A_{i'j'}$, then there is a gamemorphism $(\varphi, \rho)\colon A \to A' = (m(A_{i'j'}))$.

Proof.

$$\varphi_l = \begin{pmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_{m'} \end{pmatrix}, \qquad \varphi_r = \begin{pmatrix} 1_{n_1} & & & 0 \\ & 1_{n_2} & & \\ & & \ddots & \\ 0 & & & 1_{n_{n'}} \end{pmatrix},$$

$$\rho_l = \begin{pmatrix} 1_{m_1} & & & 0 \\ & 1_{m_2} & & \\ & & \ddots & \\ 0 & & & 1_{m_{m'}} \end{pmatrix}, \qquad \rho_r = \begin{pmatrix} q_1 & & & 0 \\ & q_2 & & \\ & & \ddots & \\ 0 & & & q_{n'} \end{pmatrix}.$$

Note. Not all gamemorphisms are of the Gale–Kuhn–Tucker type. For example,

$$\varphi_l=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}=\rho_r^t$$

and

$$\varphi_r=\begin{pmatrix}1&0&\frac{1}{2}\\0&1&\frac{1}{2}\end{pmatrix}=\rho_l^t$$

give

$$(\varphi,\rho):\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\to\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

Note. The matrices φ_r and ρ_l in the proof of the preceding theorem represent functions; that of φ_r (ρ_l) taking the columns (rows) of A to the columns (rows) of A'. In order to avoid proliferating notation we shall represent these functions by φ_r and ρ_l also. In order to facilitate the statement of the next theorem, we define a *G–K–T morphism* $(\varphi_r,\rho_l)\colon A\to A'$ to consist of a pair of functions; φ_r on the columns of A onto the columns of A', and ρ_l on the rows of A onto the rows of A'; and such that there exist sets of strategy vectors $\{p_{i'}\}_{1\leqslant i'\leqslant m'}$ and $\{q_{j'}\}_{1\leqslant j'\leqslant n'}$ which satisfy the hypothesis of the Gale–Kuhn–Tucker theorem.

DEFINITION. A G–K–T morphism $(\varphi_r,\rho_l)\colon A\to A'$ is called *positive* if the simultaneously optimal strategies $p_{i'}$, $1\leqslant i'\leqslant m'$, and $q_{j'}$, $1\leqslant j'\leqslant n'$, take positive values at every point in their respective blocks of rows or columns.

THEOREM 7. *If the G–K–T morphisms of the diagram*

$$\begin{tikzcd} & B^{(1)} \\ A \arrow[ur, "(\varphi_r^{(1)},\rho_l^{(1)})"] \arrow[dr, "(\varphi_r^{(2)},\rho_l^{(2)})"'] & \\ & B^{(2)} \end{tikzcd}$$

are positive, then it has a pushout, i.e., there exists a unique matrix D and positive G–K–T morphisms which make the diagram

$$\begin{tikzcd} & B^{(1)} \arrow[dr, "{(\delta_r^{(1)},\tau_l^{(1)})}"] & \\ A \arrow[ur, "{(\varphi_r^{(1)},\rho_l^{(1)})}"] \arrow[dr, "{(\varphi_r^{(2)},\rho_l^{(2)})}"'] & & D \\ & B^{(2)} \arrow[ur, "{(\delta_r^{(2)},\tau_l^{(2)})}"'] & \end{tikzcd}$$

commute.

Proof. Restricting to the functions on $R(A)$, the set of rows of A, we have the diagram

$$\begin{tikzcd} & R(B^{(1)}) \\ R(A) \arrow[ur, "\varphi_r^{(1)}"] \arrow[dr, "\varphi_r^{(2)}"'] & \\ & R(B^{(2)}) \end{tikzcd}$$

which will have a pushout

$$\begin{tikzcd} & R(B^{(1)}) \arrow[dr, "\delta_r^{(1)}"] & \\ R(A) \arrow[ur, "\varphi_r^{(1)}"] \arrow[dr, "\varphi_r^{(2)}"'] & & P \\ & R(B^{(2)}) \arrow[ur, "\delta_r^{(2)}"'] & \end{tikzcd}$$

in SET, the category of sets. Similarly, restricting to the functions on $C(A)$, the set of columns of A, we obtain the pushout (in SET)

$$\begin{tikzcd} & C(B^{(1)}) \arrow[dr, "\tau_l^{(1)}"] & \\ C(A) \arrow[ur, "\rho_l^{(1)}"] \arrow[dr, "\rho_l^{(2)}"'] & & Q. \\ & C(B^{(2)}) \arrow[ur, "\tau_l^{(2)}"'] & \end{tikzcd}$$

Let then, $R(D)=P$, $C(D)=Q$ and define $D=(d_{i''j''})$ by $d_{i''j''}=m[(\delta_r^{(1)}\circ\varphi_r^{(1)})^{-1}(i'')\cap(\tau_l^{(1)}\circ\rho_l^{(1)})^{-1}(j'')]$. In other words, the partition of $R(A)$ for the pushout will be the supremum of the two given partitions in the lattice of partitions ordered by unrefinement; and the partitions of $R(B^{(1)})$ and $R(B^{(2)})$ will be induced from that—similarly for the columns.

We must show then that there exist sets

$$\{p_{i''}^{(1)}\}_{1\leqslant i''\leqslant m''} \quad \text{and} \quad \{q_{j''}^{(1)}\}_{1\leqslant j''\leqslant n''}$$

of simultaneously optimal row and column strategies for the partitions of $B^{(1)}$ (and similarly for $B^{(2)}$): Given (i'',j''), let $\alpha_{i''j''}^{(1)}$ be an optimal row strategy for $B_{i''j''}^{(1)}=(\delta_r^{(1)})^{-1}(i'')\cap(\tau_l^{(1)})^{-1}(j'')$. Now $\alpha_{i''j''}^{(1)}$ may depend on j'', but we would like it not to be. By the definition of G–K–T morphism and Theorems 4 and 6, $\alpha_{i''j''}^{(1)}(\varphi_l^{(1)}|_{i''})$ is optimal for $A_{i''j''}$ and by the same reasoning $(((\alpha_{i''j''}^{(1)}\varphi_l^{(1)}|_{i''})\,\rho_l^{(2)}|_{i''})\,\varphi_l^{(2)}|_{i''})\,\rho_l^{(1)}|_{i''}=\alpha_{i''j''}^{(1)}(\varphi_l^{(1)}\rho_l^{(2)}\varphi_l^{(2)}\rho_l^{(1)}|_{i''})$ is optimal for $B_{i''j''}^{(1)}$. Since $\varphi_l^{(1)}\rho_l^{(2)}\varphi_l^{(2)}\rho_l^{(1)}|_{i''}=T_{i''}$ is nonnegative and row-stochastic, it is the transition matrix of a Markov chain. The assumption of positivity for the $(\varphi_r^{(1)},\rho_l^{(1)})$ and $(\varphi_r^{(2)},\rho_l^{(2)})$ implies that the Markov chain is irreducible. Therefore by the theorem about invariant distributions for Markov chains (see [2, Section XV.6]), $\alpha_{i''j''}^{(1)}(T_{i''})^n$ converges to a limit as n goes to ∞. This limit distribution is, by the continuity of expectation, optimal for $B_{i''j''}^{(1)}$. Also it is independent of the initial distribution and therefore independent of j''. Thus we may define $p_{i''}^{(1)}$ as the solution of

$$\begin{aligned} xT_{i''}&=x,\\ x1_{n_{i''}}&=1,\\ x&\geqslant 0. \end{aligned}$$

COROLLARY. *The category $MG/G\text{–}K\text{–}T^+$ with matrix games as objects and positive G–K–T morphisms has pushouts.*

Note. Coequalizers may be constructed for $MG/G\text{–}K\text{–}T^+$ in the same way. "Pushouts" and "coequalizers" may be extended to arbitrary $G\text{–}K\text{–}T$ morphisms by simply ignoring rows or columns which have probability zero with respect to all simultaneously optimal strategies. The quotation marks denote the fact that the resulting diagram need no longer commute.

Note. We may now interpret the Note preceding Theorem 6 as showing that every connected component of $MG/G\text{–}K\text{–}T$ has a terminal object.

EXAMPLE 5. Looking back at Examples 1, 2b, 3a and 3b, we see that the l.p. is equivalent to the game-program $P_{A',+,+}$ with

$$A = \begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & 1 & \frac{2}{3} \\ 2 & -1 & 5 & -3 \\ -1 & 2 & -3 & 5 \end{pmatrix}.$$

A' has the same symmetry as the original l.p., and it is a positive G–K–T morphism. $\iota_{A'}$ is also a positive G–K–T morphism so we can construct their coequalizers: The simultaneously optimal row strategy for both blocks of rows is $(\frac{1}{2}, \frac{1}{2})$ and for the columns

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

so that their coequalizer is

$$A'' = \begin{pmatrix} \frac{1}{6} & \frac{5}{6} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

Furthermore, the entry $\frac{1}{2}$ in A'' is a saddle point which gives a reduction to $A''' = (\frac{1}{2})$. The original l.p. is equivalent then to $P_{A''',+,+}$ and therefore $v(P) = v(P_{A''',+,+}) = 2$.

7. Conclusions and Comments

As the discerning reader can see, this paper only begins the global study of l.p.'s. Many l.p.'s which occur in combinatorial theory, e.g., the maxflow problem, have a mixture of equality and inequality constraints, so it would be desirable to unite LP and $\overline{LP}$ in a single category. The author [5] has already defined some notions of morphism for the maxflow problem which should be a special case of this general theory. To be candid, the idea of l.p.'s having nontrivial and useful notions of morphism was suggested by the global studies of the maxflow and minpath problems [6]. However, this paper was written, as much as possible, without referring to the others. This has paid off in the improvement of some technical details (e.g., compare the proofs of existence of pushouts for MG/G–K–T and FLOW in [5]) and variations in other results which point the way to deeper insight.

Even the modest goals with which this paper was begun have not been fully realized. When I started writing, I thought I possessed the means to construct pushouts and coequalizers for LP, but, after some struggle, had to settle for MG/G–K–T. The broader question still remains, as well as that for the other universal constructions. Also, the question of whether any two of LP, $\overline{LP}$ and MG are adjoint equivalent is still open.

In closing I will mention one insight which I have gained in this study: One of the questions in the catechism of category theory is "Given a 'nice' category whose morphisms are transformations, what problems (parameters) are preserved by those transformations?" In the case of $MG/G\text{-}K\text{-}T$ we observed that the components of the category each have a terminal object, (m). Thus, $m(A)$, the minimax value, is essentially the only parameter of matrices preserved by $G\text{-}K\text{-}T$ morphisms. This suggests that we might gain information about invariants for other categories such as FLOW [5] and PATH [6] by studying their local terminal objects.

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